A DIFFUSION PROBLEM WITH GRADIENT CONSTRAINT DEPENDING ON THE TEMPERATURE

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Abstract. We consider a system of a variational inequality with gradient constraint depending on the temperature, coupled with the heat equation. We prove existence of solution of this system by approximating it by a system of equations and using a fixed point argument.

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1 Introduction and main result

Variational and quasi-variational inequalities with gradient constraint appear in different situations in the literature. They model many problems such as the elastic-plastic torsion problem ([1]), sand piles and river networks ([10]), diffusion with gradient constraint ([12], [13]), superconductors ([9], [11]) or processes depending on the temperature, as the (stationary) model of the torsion problem with variable threshold of plasticity, proposed in [5]. A very interesting open question consists on proving existence of solution for the model problem proposed by Prighozin in [9], a degenerate quasi-variational inequality with a curl constraint depending on the solution. The problem studied in [11] corresponds to the nondegenerate version of this problem in a longitudinal geometry (which transforms the curl constraint in a gradient constraint). A recent paper, [8], proves existence of solution of a stationary quasi-variational inequality involving the p-curl operator and a curl constraint.

In this paper we consider the evolutive version with Dirichlet homogeneous boundary condition of the problem proposed in [5]. The problem consists on a system of a evolutive variational inequality with a gradient constraint depending on the temperature, coupled with the heat equation.

Existence of a solution for an approximated system is proved in Section 2 applying a fixed point theorem.

In Section 3, making use of appropriate *a priori* estimates on the approximated solutions it is possible to pass to the limit, proving the existence of solution of the problem. These estimates are obtained with the help of results proved in [13] as well as well known classical results (that can be found in [6], for instance).

Many interesting questions remain open, namely the existence of solution for a stronger version of this problem, the uniqueness of solution, at least for some given data and the asymptotic stabilization of the solutions when $t \to +\infty$.

We start presenting the model problem. Following [11], we consider a critical-state model of type-II superconductors with a longitudinal geometry. Here, the unknown is the magnetic field $H = (0, 0, u(x, t))$, where $x = (x_1, x_2) \in \Omega \subseteq \mathbb{R}^2$. Recall the Maxwell's equations,

$$\mu\,\partial_t H + \nabla \times E = 0, \qquad \nabla \times H = J, \tag{1}$$

where E denotes the electric field and J the density of the induced current. Here

$$J = \left(\frac{\partial u}{\partial x_2}, -\frac{\partial u}{\partial x_1}, 0\right) = \nabla^{\perp} u\ .$$

We consider an extensions of the classical Ohms law $E = \rho J$, assuming that the scalar resistivity ρ may depend on the temperature, i.e., $\rho = \rho(\theta)$. We also assume that the temperature depends on the magnetic field, satisfying the heat equation

$$\partial_t \theta - \Delta\theta = g(u),$$

being g a given function.

Supposing that the current density cannot exceed a critical value $F(\theta)$ (being F a positive function), the constitutive relation for E is assumed to be the following:

$$E = \begin{cases} \rho_0 \nabla^{\perp} u & \text{if } |\nabla u| < F(\theta), \\ (\rho_0 + \lambda)\nabla^{\perp} u & \text{if } |\nabla u| = F(\theta), \end{cases} \tag{2}$$

ρ_0 a positive constant and $\lambda \geq 0$ an unknown Lagrange multiplier.

Going back to (1) we obtain

$$\mu \partial_t u - \nabla \cdot ((\rho_0 + \lambda)\nabla u) = 0, \quad \text{where } \lambda = 0 \text{ if } |\nabla u| < F(\theta). \tag{3}$$

Defining the family of convex sets

$$\mathbb{K}_{F(\theta(t))} = \{v \in H_0^1(\Omega) : |\nabla v| \leq F(\theta(t) \text{ for a.e. } x \in \Omega\}, \tag{4}$$

multiplying equation (3) by $v - u(t)$, being $v \in \mathbb{K}_{F(\theta(t))}$, we obtain, integrating over Ω,

$$\int_\Omega \mu \partial_t u(t)(v - u(t)) + \int_\Omega (\rho_0 + \lambda)\nabla u(t) \cdot \nabla(v - u(t)) = 0.$$

As $\lambda \nabla u(t) \cdot \nabla(v - u(t)) \leq \lambda |\nabla u(t)| \, (|\nabla v| - |\nabla u(t)|) \leq 0$, we have

$$\int_\Omega \mu \partial_t u(t)(v - u(t)) + \int_\Omega \rho_0 \nabla u(t) \cdot \nabla(v - u(t)) \geq 0.$$

Impose initial conditions $u(0) = u_0$ and $\theta(0) = \theta_0$ and homogeneous Dirichlet boundary conditions, assuming that Ω is a bounded domain of $\mathbb{R}^2$.

For the mathematical point of view there exists no additional difficulty is we assume that Ω if a bounded open subset of $\mathbb{R}^N$. We also assume that the boundary $\partial\Omega$ is of class $C^{2,\alpha}$, where $\alpha = 1 - \frac{N}{q}$ with $q > N$. In what follows, T denotes a positive number, I the interval $[0, T]$, Q_T the cylinder $\Omega \times]0, T[$, Σ the lateral boundary $\partial\Omega \times I$ of Q_T. Here, ∇ denotes the spatial gradient, i.e., $\nabla = (\partial_{x_1}, \ldots, \partial_{x_N})$.

Assuming that the independent term of (1) is a given function f (not necessarily the null function) and taking, for simplicity, $\mu = \rho_0$, we are now led to the following coupled system

$$\begin{cases} u \in \ L^2(0, T; H_0^1(\Omega)), \ \partial u \in L^2(0, T; H^{-1}(\Omega)), \\ u(t) \in \mathbb{K}_{F(\theta(t))} \text{ for a.e. } t \in]0, T[, \qquad u(0) = u_0, \\ \displaystyle\int_\Omega \partial_t u(t)(v - u(t)) + \int_\Omega \nabla u(t) \cdot \nabla(v - u(t)) \ \geq \ \int_{Q_T} f(t)(v - u(t)), \\ \qquad\qquad\qquad\qquad\qquad\qquad \forall\, v \in \mathbb{K}_{F(\theta(t))}, \text{ for a.e. } t \in]0, T[; \end{cases} \tag{5}$$

$$\begin{cases} \partial_t \theta - \triangle\, \theta & = & g(u) \quad \text{in } Q_T \\ \theta(0) & = & \theta_0 \qquad \text{on } \Omega, \quad \theta_{|\Sigma} = 0. \end{cases} \tag{6}$$

Here, the term $\int_\Omega \partial_t u(t)(v-u(t))$ is interpreted in the duality between $L^2(0,T;H_0^1(\Omega))$ and $L^2(0,T;H^{-1}(\Omega))$. Note that, since u belongs to $L^2(0,T;H_0^1(\Omega))$ and $\partial_t u$ belongs to $L^2(0,T;H^{-1}(\Omega))$ we have $u\in C([0,T];L^2(\Omega))$ and so $u(0)$ has a meaning.

We call *strong variational formulation of the problem* to the system (5)-(6). The variational inequality (5) is the usual formulation of a evolutive variational inequality, when its solutions are continuous in the variable t. It is not always possible to prove the existence of solutions with this regularity in the temporal variable. To deal with those situations, Lions introduced a weaker version for evolutive variational inequalities, as it can be found in [7], pages 266-269. In our framework, letting

$$\mathcal{K}_{F(\theta)} \;=\; \{v\in L^2(0,T;H_0^1(\Omega)) : |\nabla v|\le F(\theta) \text{ a.e. in } Q_T\},$$

it corresponds to solve the system (7)-(6), where

$$\begin{cases} v\in L^2(0,T;H_0^1(\Omega)), \qquad \partial_t v\in L^2\left(0,T;H^{-1}(\Omega)\right), \qquad v(0)=u_0, \\[2ex] \displaystyle\int_{Q_T}\partial_t v(v-u)+\int_{Q_T}\nabla u\cdot\nabla(v-u) \;\ge\; \int_{Q_T} f(v-u), \qquad \forall\, v\in\mathcal{K}_{F(\theta)}. \end{cases} \tag{7}$$

In the literature this is called the *weak variational formulation of the problem.* Notice that, if u solves (7)-(6) and $u\in L^2(0,T;H_0^1(\Omega))$, $\partial_t u\in L^2(0,T;H^{-1}(\Omega))$ and, in addition, $u(0)=u_0$, then u solves (5)-(6). In fact, take $v=u+s(w-u)$, with $s\in]0,1[$, for test function in (7), being w an arbitrary function of $\mathcal{K}_{F(\theta)}$ such that $\partial_t w\in L^2(0,T;H^{-1}(\Omega)$. Then

$$s\int_{Q_T}\partial_t(u+s(w-u))(w-u)+s\int_{Q_T}\nabla u\cdot\nabla(w-u)\;\ge\; s\int_{Q_T} f(w-u).$$

Dividing both members of the inequality by s and letting $s\to 0$, we have

$$\int_{Q_T}\partial_t u(w-u)+\int_{Q_T}\nabla u\cdot\nabla(w-u)\;\ge\;\int_{Q_T} f(w-u), \qquad \forall\, w\in\mathcal{K}_{F(\theta)}.$$

Given $t_0\in]0,T[$ and δ a positive small number, let v be such that $v(t)\in\mathbb{K}_{F(\theta(t))}$, for a.e. $t\in]t_0-\delta,t_0+\delta[$ and define

$$w(t)=\begin{cases} v(t) & \text{if } t\in]t_0-\delta,t_0+\delta[\\ u(t) & \text{otherwise.}\end{cases}$$

Then,

$$\int_{t_0-\delta}^{t_0+\delta}\int_\Omega \partial_t u(v-u)+\int_{t_0-\delta}^{t_0+\delta}\int_\Omega \nabla u\cdot\nabla(v-u)\;\ge\;\int_{t_0-\delta}^{t_0+\delta}\int_\Omega f(v-u),$$

and dividing both members by 2δ and letting $\delta\to 0$, we get, for a.e. $t\in]0,T[$,

$$\int_\Omega \partial_t u(t)(v-u(t))+\int_\Omega \nabla u(t)\cdot\nabla(v-u(t))\;\ge\;\int_\Omega f(t)(v-u(t)).$$

In this paper, we prove that the problem (7)-(6) has a solution. We are not able to prove existence of solution for the strong formulation of the problem, but we prove existence of solution of the *intermediate variational formulation of the problem* (8)-(6) which also solves the *weak variational formulation of the problem.*

But first we introduce some notations related to the Sobolev spaces used along the paper. The notations related to the subscripts and superscripts in these spaces, following the notations of [6] and [7], will be clear from the examples: for $p \in [1, \infty]$,

$$W_p^1(\Omega) = \{u \in L^p(\Omega) : \nabla u \in [L^p(\Omega)]^N\};$$

$$W_{p,0}^1(\Omega) = \{u \in W_p^1(\Omega) : u_{|\partial\Omega} = 0\};$$
$$W_p^{2,1}(Q_T) = W_p^1(0,T; L^p(\Omega)) \cap L^p(0,T; W_p^2(\Omega)).$$

The *intermediate variational formulation of the problem* consists of the coupled system of (6) with the following inequality:

$$\begin{cases} u(t) \in \mathbb{K}_{F(\theta(t))} \text{ for a.e. } t \in]0,T[, \\ \partial_t u \in \big(L^\infty(0,T; W_{\infty,0}^1(\Omega))\big)' \\ \displaystyle\int_{Q_T} \partial_t u(v-u) + \int_{Q_T} \nabla u \cdot \nabla(v-u) \geq \int_{Q_T} f(v-u), \\ \forall v \in C([0,T]; L^2(\Omega)) : v(0) = u_0, \quad v(t) \in \mathbb{K}_{F(\theta(t))}, \text{ for a.e. } t \in]0,T[. \end{cases} \tag{8}$$

Calling

$$X = L^\infty(0,T; W_{\infty,0}^1(\Omega)),$$

the integral $\displaystyle\int_{Q_T} \partial_t u(v-u)$ is interpreted in the duality between X and X'.

Considering the following assumptions on the data,

$$\begin{cases} F \in C(\mathbb{R}) : \qquad F > 0 \\ f \in L^\infty(Q_T) \\ u_0 \in H_0^1(\Omega), \ |\nabla u_0| \leq F(\theta_0), \end{cases} \tag{9}$$

and

$$\begin{cases} g \in C(\mathbb{R}) \\ \theta_0 \in W_q^2(\Omega) \cap W_{\infty,0}^1(\Omega), \qquad (q > N), \end{cases} \tag{10}$$

we will prove in Section 3 the main result of this paper:

Theorem 1.1. *With the assumptions* (9)-(10) *the problem* (8)-(6) *has a solution*

$$(u,\theta) \in L^\infty(0,T; W_{\infty,0}^1(\Omega)) \times W_q^{2,1}(Q_T) \quad \textit{with} \quad \partial_t u \in \big(L^\infty(0,T; W_{\infty,0}^1(\Omega))\big)'$$

that also solves problem (7)-(6).

2 The approximating problems

We start considering the following family of approximating (with $\varepsilon > 0$) systems of equations

$$\left\{\begin{array}{l}\left\{\begin{array}{ll}\partial_t u^\varepsilon - \nabla\cdot(k_\varepsilon(|\nabla u^\varepsilon|^2 - F_\varepsilon^2(\theta^\varepsilon))\nabla u^\varepsilon) = f^\varepsilon & \text{in } Q_T,\\ u^\varepsilon(0) = u_0^\varepsilon \ \text{ in } \Omega, \qquad u^\varepsilon = 0 \ \text{ on } \Sigma, & \end{array}\right.\\ \\ \left\{\begin{array}{ll}\partial_t\theta^\varepsilon - \Delta\theta^\varepsilon = g^\varepsilon(u^\varepsilon) \qquad \text{in } Q_T, & \\ \theta^\varepsilon(0) = \theta_0^\varepsilon \ \text{ in } \Omega,\ \theta^\varepsilon = 0 \qquad \text{on } \Sigma, & \end{array}\right.\end{array}\right. \tag{11}$$

where

- $k_\varepsilon : \mathbb{R} \longrightarrow \mathbb{R}$ is a C^2 nondecreasing function such that $k_\varepsilon(s) = 1$ if $s \leq 0$, $k_\varepsilon(s) = e^{s/\varepsilon}$ if $s \geq \varepsilon$;
- F_ε is a C^∞ approximation of F in $C(\mathbb{R})$;
- f_ε a C^∞ approximation of f in $L^{q\vee 2}(Q_T)$, satisfying $f_\varepsilon(x, 0) = 0$, for $x \in \partial\Omega$;
- g_ε a C^∞ approximation of g in $L^{q\vee 2}(Q_T)$, verifying $g^\varepsilon(0) = 0$;
- $u_0^\varepsilon, \theta_0^\varepsilon \in \mathcal{D}(\Omega)$, θ_0^ε an approximation of θ_0 in $W_q^2(\Omega) \cap W_{\infty,0}^1(\Omega)$ and u_0^ε an approximation of u_0 in $H_0^1(\Omega)$, verifying $|\nabla u_0^\varepsilon| \leq F^\varepsilon(\theta_0^\varepsilon)$.

The regularization introduced by the function k_ε follows an idea of [3]. As k_ε is an increasing positive function then,

$$\forall\, X, Y \in \mathbb{R}^n\ \forall\, a \in \mathbb{R} \quad \Big(k_\varepsilon\left(|X|^2 - a\right)\, X - k_\varepsilon\left(|Y|^2 - a\right)\, Y\Big)\cdot(X - Y) \geq 0. \tag{12}$$

We will prove that system the (11) has a solution in $C^{2,1}_{\alpha,\alpha/2}(\overline{Q}_T) \times C^{2,1}_{\alpha,\alpha/2}(\overline{Q}_T)$. For this purpose we need some auxiliary results.

Proposition 2.1. *For every $\theta \in C^{1,0}_{\alpha,\alpha/2}(\overline{Q}_T)$, the problem*

$$\left\{\begin{array}{ll}\partial_t u^\varepsilon - \nabla\cdot(k_\varepsilon(|\nabla u^\varepsilon|^2 - F_\varepsilon^2(\theta))\nabla u^\varepsilon) = f_\varepsilon & \textit{in } Q_T,\\ u^\varepsilon(0) = u_0^\varepsilon \ \textit{ in } \Omega, \qquad u^\varepsilon = 0 \ \textit{ on } \Sigma, & \end{array}\right. \tag{13}$$

has a unique solution $u^\varepsilon \in C^{2,1}_{\alpha,\alpha/2}(\overline{Q}_T)$ and

$$\exists\, M > 0 \ \ \forall\, \varepsilon \in]0,1[\ \ \forall\, \theta \in C^{1,0}_{\alpha,\alpha/2}(\overline{Q}_T) \qquad \|u^\varepsilon\|_{L^\infty(Q_T)} \leq M. \tag{14}$$

Proof. The existence of solution for this problem, as well as the uniform boundedeness of the solution in $L^\infty(Q_T)$ are direct consequences of the general parabolic theory for quasilinear non-degenerate equations ([4, 6]). □

Proposition 2.2. *For $u \in C_{\alpha,\alpha/2}(\overline{Q}_T)$ such that $u_{|\Sigma} = 0$, with the assumptions* (10), *the problem*

$$\begin{cases} \partial_t \theta^\varepsilon - \Delta \theta^\varepsilon = g^\varepsilon(u) & \text{in } Q_T, \\ \theta^\varepsilon(0) = \theta_0^\varepsilon \quad \text{in } \Omega, & \theta^\varepsilon = 0 \quad \text{on } \Sigma. \end{cases}$$

has a unique solution $\theta^\varepsilon \in C^{2,1}_{\alpha,\alpha/2}(\overline{Q}_T)$ and

$$\exists\, C > 0 \quad \forall\, \varepsilon \in]0,1[\qquad \|\theta^\varepsilon\|_{C^{2,1}_{\alpha,\alpha/2}(\overline{Q}_T)} \leq C\big(\|g^\varepsilon(u)\|_{C_{\alpha,\alpha/2}(\overline{Q}_T)} + \|\theta_0^\varepsilon\|_{C^2_\alpha(\overline{\Omega})}\big) \tag{15}$$

and

$$\exists\, C > 0 \quad \forall\, \varepsilon \in]0,1[\qquad \|\theta^\varepsilon\|_{W^{2,1}_q(Q_T)} \leq C\big(\|g^\varepsilon(u)\|_{L^q(Q_T)} + \|\theta_0^\varepsilon\|_{W^2_q(\Omega)}\big). \tag{16}$$

Proof. To prove (15), just apply Theorem 5.2, page 320 of [6], noting that $g^\varepsilon \circ u \in C_{\alpha,\alpha/2}(\overline{Q}_T)$. The proof of (16) is a direct consequence of Theorem 9.1, page 341 of [6]. $\square$

Consider, the diagram

$$\begin{tikzcd} C^{1,0}_{\alpha,\alpha/2}(\overline{Q}_T) \arrow[r, "i \circ \Psi_\varepsilon \circ \Phi_\varepsilon"] \arrow[d, "\Phi_\varepsilon"'] & C^{1,0}_{\alpha,\alpha/2}(\overline{Q}_T) \\ C_{\alpha,\alpha/2}(\overline{Q}_T) \arrow[r, "\Psi_\varepsilon"] & W^{2,1}_q(Q_T) \arrow[u, "i"'] \end{tikzcd}$$

where, for $\theta \in C^{1,0}_{\alpha,\alpha/2}(\overline{Q}_T)$ and $u \in C_{\alpha,\alpha/2}(\overline{Q}_T)$, $\Phi_\varepsilon(\theta)$ and $\Psi_\varepsilon(u)$ are given by the Propositions 2.1 and 2.2 respectively, and i is the (compact) inclusion of $W^{2,1}_q(Q_T)$ into $C^{1,0}_{\alpha,\alpha/2}(\overline{Q}_T)$.

We will prove that the function $G_\varepsilon = i \circ \Psi_\varepsilon \circ \Phi_\varepsilon$ is continuous and compact.

Proposition 2.3. *With the assumption* (9), *the function Φ_ε is continuous.*

Proof. Let $(\theta_n)_n$ be a sequence in $C^{1,0}_{\alpha,\alpha/2}(\overline{Q}_T)$ converging, in this space to a function θ. Denote $u_n^\varepsilon = \Phi_\varepsilon(\theta_n)$ and $u^\varepsilon = \Phi_\varepsilon(\theta)$. By applying Theorem 9.1, page 341 of [6], we know that he sequence $\{u_n^\varepsilon\}_n$ is bounded in $W^{2,1}_q(Q_T)$, since $F^\varepsilon(\theta_n) \xrightarrow[n]{} F(\theta)$ in $C^{1,0}_{\alpha,\alpha/2}(\overline{Q}_T)$. The compact inclusion of $W^{2,1}_q(Q_T)$ into $C^{1,0}_{\alpha,\alpha/2}(\overline{Q}_T)$ implies that this sequence belongs to a compact subset of $C_{\alpha,\alpha/2}(\overline{Q}_T)$. As $C_{\alpha,\alpha/2}(\overline{Q}_T)$ is continuously included in $L^\infty(0,T;L^2(\Omega))$ we only need to prove that $(u_n^\varepsilon)_n$ converges to u^ε in $L^\infty(0,T;L^2(\Omega))$.

Multiplying the equations that define u_n^ε and u^ε both by $u_n^\varepsilon - u^\varepsilon$, subtracting one equation from the other and integrating over $Q_t = \Omega \times]0,t[$ we obtain

$$\begin{aligned} &\frac{1}{2}\int_\Omega |u_n^\varepsilon(t) - u^\varepsilon(t)|^2 \\ &\quad + \int_{Q_t} \Big[k_\varepsilon\left(|\nabla u_n^\varepsilon|^2 - F_\varepsilon^2(\theta_n)\right)\nabla u_n^\varepsilon - k_\varepsilon\left(|\nabla u^\varepsilon|^2 - F_\varepsilon^2(\theta_n)\right)\nabla u^\varepsilon\Big] \cdot \nabla(u_n^\varepsilon - u^\varepsilon) \\ &\qquad = \int_{Q_t} \big[k_\varepsilon(|\nabla u^\varepsilon|^2 - F^2(\theta)) - k_\varepsilon(|\nabla u^\varepsilon|^2 - F^2(\theta_n))\big]\nabla u^\varepsilon \cdot \nabla(u_n^\varepsilon - u^\varepsilon) \end{aligned}$$

Observe that

- $\displaystyle\int_{Q_t} \Big[k_\varepsilon(|\nabla u_n^\varepsilon|^2 - F_\varepsilon^2(\theta_n))\nabla u_n^\varepsilon - k_\varepsilon(|\nabla u^\varepsilon|^2 - F_\varepsilon^2(\theta_n))\nabla u^\varepsilon\Big] \cdot \nabla(u_n^\varepsilon - u^\varepsilon) \geq 0$, by (12);
- ∇u_n^ε is bounded in $L^\infty(Q_T)$ independently of n (recall the uniform boundedeness of u_n^ε in $W_q^{2,1}(Q_T)$ referred at the beginning of this proof);
- $\theta_n(x,t) \xrightarrow[n]{} \theta(x,t)$ for every $(x,t) \in Q_T$.

As a consequence we get

$$\int_\Omega |u_n^\varepsilon(t) - u^\varepsilon(t)|^2$$
$$\leq 2\int_{Q_T} \Big[k_\varepsilon(|\nabla u^\varepsilon|^2 - F^2(\theta)) - k_\varepsilon(|\nabla u^\varepsilon|^2 - F^2(\theta_n))\Big]\nabla u^\varepsilon \cdot \nabla(u_n^\varepsilon - u^\varepsilon) \xrightarrow[n]{} 0,$$

and so $(u_n^\varepsilon)_n$ converges to u^ε in $L^\infty(0,T;L^2(\Omega))$. □

Proposition 2.4. *With the assumption* (10) *the function* Ψ_ε *is continuous.*

Proof. If $u_1, u_2 \in C_{\alpha,\alpha/2}(\overline{Q}_T)$ let $\theta_1^\varepsilon = \Psi_\varepsilon(u_1)$ and $\theta_2^\varepsilon = \Psi_\varepsilon(u_2)$. Then $\theta_1^\varepsilon - \theta_2^\varepsilon$ is the solution of the problem

$$\begin{cases} \partial_t\theta - \Delta\theta = g^\varepsilon(u_1) - g^\varepsilon(u_2) & \text{in } Q_T, \\ \theta(0) = 0 \;\text{ in } \Omega, \qquad \theta = 0 \;\text{ on } \Sigma. \end{cases}$$

and then, as in Proposition 2.2, there exists $C > 0$ such that

$$\|\theta_1^\varepsilon - \theta_2^\varepsilon\|_{C^{2,1}_{\alpha,\alpha/2}(\overline{Q}_T)} \;\leq\; C\,\|g^\varepsilon(u_1) - g^\varepsilon(u_2)\|_{C_{\alpha,\alpha/2}(\overline{Q}_T)}.$$

To conclude the proof we use the continuous inclusion of $C^{2,1}_{\alpha,\alpha/2}(\overline{Q}_T)$ into $W_q^{2,1}(Q_T)$. □

Proposition 2.5. *With the assumption* (9) *and* (10), $\Psi^\varepsilon \circ \Phi^\varepsilon$ *is bounded. More precisely,*

$$\exists N > 0 \;\; \forall\, \varepsilon \in]0,1[\;\; \forall\, \theta \in C^{1,0}_{\alpha,\alpha/2}(\overline{Q}_T) \quad \|\Psi^\varepsilon(\Phi^\varepsilon(\theta))\|_{W_q^{2,1}(Q_T)} \leq N. \tag{17}$$

Proof. By (14), consider $M > 0$ such that, for all $\theta \in C^{1,0}_{\alpha,\alpha/2}(\overline{Q}_T)$, $\|\Phi^\varepsilon(\theta)\|_{L^\infty(Q_T)} \leq M$. In particular, $\|g^\varepsilon(\Phi^\varepsilon(\theta))\|^q_{L^q(Q_T)} \leq |Q_T| \max\limits_{[-M,M]} |g|$. The conclusion now follows from (16). □

Theorem 2.6. *The system* (11) *has a solution*

$$(u^\varepsilon, \theta^\varepsilon) \in C^{2,1}_{\alpha,\alpha/2}(\overline{Q}_T) \times C^{2,1}_{\alpha,\alpha/2}(\overline{Q}_T).$$

Proof. Using the previous propositions and the compact inclusion of $W_q^{2,1}(Q_T)$ into $C^{1,0}_{\alpha,\alpha/2}(\overline{Q}_T)$ we can apply Schauder's Fixed Point Theorem to the function G_ε, obtaining a fixed point $(u^\varepsilon, \theta^\varepsilon)$ for G_ε. In particular, as $\theta^\varepsilon = \Psi_\varepsilon(\Phi_\varepsilon(\theta^\varepsilon))$, we have $\theta^\varepsilon \in C^{2,1}_{\alpha,\alpha/2}(\overline{Q}_T)$ and, using the Proposition 2.1, we also have $u^\varepsilon \in C^{2,1}_{\alpha,\alpha/2}(\overline{Q}_T)$. □

3 Existence of solution

We start with some preliminary norm bounds (independently of ε) that will be needed in the sequel. In the following, C will denote the Poincaré constant in $H_0^1(\Omega)$.

Lemma 3.1. *Assume* (9)-(10)*. Let* $\{(u^\varepsilon, \theta^\varepsilon)\}_{0<\varepsilon<1}$ *be a family of solutions of the problem* (11)*. Then, there exist* m*,* M*,* C_1*,* C_2*,* C_3 *and* D_p*, for* $p \geq 1$*, such that, for* ε *small enough:*

a) $0 < m \leq \|F_\varepsilon(\theta^\varepsilon)\|_{L^\infty(Q_T)} \leq M$;

b) $\|\nabla u^\varepsilon\|^2_{L^2(Q_T)} \leq \|k_\varepsilon(|\nabla u^\varepsilon|^2 - F_\varepsilon^2(\theta^\varepsilon))\,|\nabla u^\varepsilon|^2\|_{L^1(Q_T)} \leq C_1$;

c) $\|k_\varepsilon(|\nabla u^\varepsilon|^2 - F_\varepsilon^2(\theta^\varepsilon))\|_{L^1(Q_T)} \leq C_2$;

d) $\|k_\varepsilon(|\nabla u^\varepsilon|^2 - F_\varepsilon^2(\theta^\varepsilon))\,|\nabla u^\varepsilon|\|_{L^1(Q_T)} \leq C_3$;

e) $\|\nabla u^\varepsilon\|_{L^p(Q_T)} \leq D_p$.

Proof.
a) Recall that $\{\theta^\varepsilon\}_{0<\varepsilon<1}$ is a bounded subset of $C(\overline{Q}_T)$, $F_\varepsilon \xrightarrow[\varepsilon\to 0]{} F$ in $C(\mathbb{R})$ and F is continuous and positive.
b) The first inequality if trivial because $k_\varepsilon \geq 1$. To prove the other inequality, we multiply the equation (13) by u^ε and integrate over Q_T. We obtain

$$\int_\Omega [u^\varepsilon(T)]^2 + \int_{Q_T} k_\varepsilon(|\nabla u^\varepsilon|^2 - F_\varepsilon^2(\theta^\varepsilon))|\nabla u^\varepsilon|^2 = \int_{Q_T} f^\varepsilon u^\varepsilon + \int_\Omega (u_0^\varepsilon)^2,$$

and then, using Hölder, Poincaré and Young inequalities,

$$\begin{aligned}
\int_{Q_T} k_\varepsilon(|\nabla u^\varepsilon|^2 - F_\varepsilon^2(\theta^\varepsilon))|\nabla u^\varepsilon|^2 &\leq \|f^\varepsilon\|_{L^2(Q_T)}\,\|u^\varepsilon\|_{L^2(Q_T)} + \|u_0^\varepsilon\|^2_{L^2(\Omega)} \\
&\leq C\|f^\varepsilon\|_{L^2(Q_T)}\,\|\nabla u^\varepsilon\|_{L^2(Q_T)} + \|u_0^\varepsilon\|^2_{L^2(\Omega)} \\
&\leq \frac{C^2}{2}\|f^\varepsilon\|^2_{L^2(Q_T)} + \frac{1}{2}\|\nabla u^\varepsilon\|^2_{L^2(Q_T)} + \|u_0^\varepsilon\|^2_{L^2(\Omega)}.
\end{aligned}$$

As $\|\nabla u^\varepsilon\|^2_{L^2(Q_T)} \leq \displaystyle\int_{Q_T} k_\varepsilon(|\nabla u^\varepsilon|^2 - F_\varepsilon^2(\theta^\varepsilon))|\nabla u^\varepsilon|^2$ we obtain

$$\int_{Q_T} k_\varepsilon(|\nabla u^\varepsilon|^2 - F_\varepsilon^2(\theta^\varepsilon))|\nabla u^\varepsilon|^2 \leq C^2\|f^\varepsilon\|^2_{L^2(Q_T)} + 2\|u_0^\varepsilon\|^2_{L^2(\Omega)},$$

which completes the proof, as $\lim\limits_{\varepsilon\to 0}\|f^\varepsilon\|_{L^2(Q_T)} = \|f\|_{L^2(Q_T)}$ and $\lim\limits_{\varepsilon\to 0}\|u_0^\varepsilon\|_{L^2(Q_T)} = \|u_0\|_{L^2(Q_T)}$.

c) We have

$$\begin{aligned}
\|k_\varepsilon(|\nabla u^\varepsilon|^2 - F_\varepsilon^2(\theta^\varepsilon))\|_{L^1(Q_T)} &= \int_{Q_T} k_\varepsilon(|\nabla u^\varepsilon|^2 - F_\varepsilon^2(\theta^\varepsilon)) \\
&\leq \frac{1}{m^2}\int_{Q_T} k_\varepsilon(|\nabla u^\varepsilon|^2 - F_\varepsilon^2(\theta^\varepsilon))\, F_\varepsilon^2(\theta^\varepsilon) \quad \text{by a)} \\
&\leq \frac{1}{m^2}\Big[\int_{Q_T} F_\varepsilon^2(\theta^\varepsilon) + \int_{Q_T} k_\varepsilon(|\nabla u^\varepsilon|^2 - F_\varepsilon^2(\theta^\varepsilon))|\nabla u^\varepsilon|^2\Big] \\
&\quad (\text{as } k_\varepsilon(a-b)\, b \leq b + k_\varepsilon(a-b)\, a \text{ for } a, b \geq 0) \\
&\leq \frac{1}{m^2}\Big[|Q_T|\, M^2 + C_1\Big] \quad \text{by a) and b).}
\end{aligned}$$

d) As

- $k_\varepsilon(|\nabla u^\varepsilon|^2 - F_\varepsilon^2(\theta^\varepsilon))\, |\nabla u^\varepsilon| = \Big[k_\varepsilon^{\frac{1}{2}}(|\nabla u^\varepsilon|^2 - F_\varepsilon^2(\theta^\varepsilon))\Big] \cdot \Big[k_\varepsilon^{\frac{1}{2}}(|\nabla u^\varepsilon|^2 - F_\varepsilon^2(\theta^\varepsilon))\, |\nabla u^\varepsilon|\Big]$,
- $\|k_\varepsilon^{\frac{1}{2}}(|\nabla u^\varepsilon|^2 - F_\varepsilon^2(\theta^\varepsilon))\|_{L^2(Q_T)} = \big(\|k_\varepsilon(|\nabla u^\varepsilon|^2 - F_\varepsilon^2(\theta^\varepsilon))\|_{L^1(Q_T)}\big)^{\frac{1}{2}}$,
- $\|k_\varepsilon^{\frac{1}{2}}(|\nabla u^\varepsilon|^2 - F_\varepsilon^2(\theta^\varepsilon))\, |\nabla u^\varepsilon|\|_{L^2(Q_T)} = \big(\|k_\varepsilon(|\nabla u^\varepsilon|^2 - F_\varepsilon^2(\theta^\varepsilon))\, |\nabla u^\varepsilon|^2\|_{L^1(Q_T)}\big)^{\frac{1}{2}}$,

we have, using Hölder inequality,

$$\|k_\varepsilon(|\nabla u^\varepsilon|^2 - F_\varepsilon^2(\theta^\varepsilon))\, |\nabla u^\varepsilon|\|_{L^1(Q_T)} \leq \sqrt{C_2 C_1}.$$

e) Let

$$Q_T^\varepsilon = \big\{(x,t) \in Q_T | \nabla u^\varepsilon(x,t)|^2 > F_\varepsilon^2(\theta^\varepsilon(x,t)) + \varepsilon\big\}. \tag{18}$$

As

$$\begin{aligned}
\|\nabla u^\varepsilon\|_{L^p(Q_T)}^p &= \|\nabla u^\varepsilon\|_{L^p(Q_T^\varepsilon)}^p + \|\nabla u^\varepsilon\|_{L^p(Q_T \setminus Q_T^\varepsilon)}^p \\
&\leq \int_{Q_T^\varepsilon} |\nabla u^\varepsilon|^p + \int_{Q_T} \big(F_\varepsilon^2(\theta^\varepsilon) + 1\big)^{\frac{p}{2}} \\
&\leq \|\nabla u^\varepsilon\|_{L^p(Q_T^\varepsilon)}^p + |Q_T|\, \big(M^2+1\big)^{\frac{p}{2}}, \text{ by a)} \\
&= \Big(\||\nabla u^\varepsilon|^2\|_{L^{\frac{p}{2}}(Q_T^\varepsilon)}\Big)^{\frac{p}{2}} + |Q_T|\, \big(M^2+1\big)^{\frac{p}{2}} \\
&\leq \Big(\||\nabla u^\varepsilon|^2 - F_\varepsilon^2(\theta^\varepsilon)\|_{L^{\frac{p}{2}}(Q_T^\varepsilon)} + \|F_\varepsilon^2(\theta^\varepsilon)\|_{L^{\frac{p}{2}}(Q_T^\varepsilon)}\Big)^{\frac{p}{2}} + |Q_T|\, \big(M^2+1\big)^{\frac{p}{2}} \\
&\leq \Big(\||\nabla u^\varepsilon|^2 - F_\varepsilon^2(\theta^\varepsilon)\|_{L^{\frac{p}{2}}(Q_T^\varepsilon)} + |Q_T|^{\frac{2}{p}}\, M^2\Big)^{\frac{p}{2}} + |Q_T|\, \big(M^2+1\big)^{\frac{p}{2}},
\end{aligned}$$

we only need to obtain an upper bound to $\||\nabla u^\varepsilon|^2 - F_\varepsilon^2(\theta^\varepsilon)\|_{L^{\frac{p}{2}}(Q_T^\varepsilon)}$.

Let $A(p) \in \mathbb{R}$ be such that $x^{\frac{p}{2}} \leq A(p) e^x$ for $x > 0$. Then

$$
\begin{aligned}
\||\nabla u^\varepsilon|^2 - F_\varepsilon^2(\theta^\varepsilon)\|^{\frac{p}{2}}_{L^{\frac{p}{2}}(Q_T^\varepsilon)} &= \int_{Q_T^\varepsilon} \left(|\nabla u^\varepsilon|^2 - F_\varepsilon^2(\theta^\varepsilon)\right)^{\frac{p}{2}} \\
&\leq \int_{Q_T^\varepsilon} \left(\frac{|\nabla u^\varepsilon|^2 - F_\varepsilon^2(\theta^\varepsilon)}{\varepsilon}\right)^{\frac{p}{2}} \\
&\leq \int_{Q_T^\varepsilon} A(p)\, e^{\frac{|\nabla u^\varepsilon|^2 - F_\varepsilon^2(\theta^\varepsilon)}{\varepsilon}} \\
&= \int_{Q_T^\varepsilon} A(p)\, k_\varepsilon \left(|\nabla u^\varepsilon|^2 - F^2(\theta^\varepsilon)\right) \\
&\leq A(p)\, \|k_\varepsilon(|\nabla u^\varepsilon|^2 - F_\varepsilon^2(\theta^\varepsilon))\|_{L^1(Q_T)}.
\end{aligned}
$$

The conclusion follows now from c). □

We are now in conditions to prove the result stated in Section 1, page 5.

Proof of Theorem 1.1. Let $\{(u^\varepsilon, \theta^\varepsilon))\}_{0<\varepsilon<1}$ be a family of solutions of the problem (11). By Lemma 3.1 and inequality (17), $\{u^\varepsilon\}_{0<\varepsilon<1}$ and $\{\theta^\varepsilon\}_{0<\varepsilon<1}$ are bounded in $L^q(0,T;W_q^1(\Omega))$ and $W_q^{2,1}(Q_T)$ respectively. Then we may assume that there exists $u \in L^q(0,T;W_q^1(\Omega))$ and $\theta \in W_q^{2,1}(Q_T)$ such that

$$
\begin{cases}
u^\varepsilon \rightharpoonup u & \text{in } L^q(0,T;W_q^1(\Omega))\text{-weak} \\
\theta^\varepsilon \rightharpoonup \theta & \text{in } W_q^{2,1}(Q_T)\text{-weak.}
\end{cases}
$$

Note that, as $\theta^\varepsilon \longrightarrow \theta$ in $C(Q_T)$, F is uniformly continuous in the range of $\{\theta^\varepsilon\}_{0<\varepsilon<1}$ and $F^\varepsilon \longrightarrow F$ in $C(K)$, where K is a compact of $\mathbb{R}$ containing the ranges of θ^ε and θ, we can conclude that

$$
F^\varepsilon(\theta^\varepsilon) \xrightarrow[\varepsilon\to 0]{} F(\theta) \quad \text{in } C(\overline{Q}_T).
$$

It is obvious that θ satisfies (6). We will prove that (u,θ) satisfies (8) in three steps.

Step 1: $u(t) \in \mathbb{K}_{F(\theta(t))}$ for a.e. $t \in]0,T[$. By the definition of k_ε, by Lemma 3.1 c) and recalling the definition of $Q_T^{\sqrt{\varepsilon}} = \{(x,t) \in Q_T : |\nabla u^\varepsilon|^2 - F_\varepsilon^2(\theta^\varepsilon) \geq \sqrt{\varepsilon}\}$, we have,

$$
\left|Q_T^{\sqrt{\varepsilon}}\right| = \int_{Q_T^{\sqrt{\varepsilon}}} 1 \leq \int_{Q_T^{\sqrt{\varepsilon}}} k_\varepsilon(|\nabla u^\varepsilon|^2 - F_\varepsilon^2(\theta^\varepsilon))\, e^{-\frac{1}{\sqrt{\varepsilon}}} \leq C_1 e^{-\frac{1}{\sqrt{\varepsilon}}}.
$$

and then, using Lemma 3.1 a) and d),

$$\begin{aligned}
\int_{Q_T} \left(|\nabla u|^2 - F^2(\theta)\right)^+ &\leq \liminf_{\varepsilon\to 0} \int_{Q_T} \left(|\nabla u^\varepsilon|^2 - F_\varepsilon^2(\theta^\varepsilon) - \sqrt{\varepsilon}\right)^+ \\
&= \liminf_{\varepsilon\to 0} \int_{Q_T^{\sqrt{\varepsilon}}} \left(|\nabla u^\varepsilon|^2 - F_\varepsilon^2(\theta^\varepsilon) - \sqrt{\varepsilon}\right) \\
&\leq \liminf_{\varepsilon\to 0} \int_{Q_T^{\sqrt{\varepsilon}}} |\nabla u^\varepsilon|^2 \\
&\leq \liminf_{\varepsilon\to 0} \| |\nabla u^\varepsilon|^2 \|_{L^2(Q_T^{\sqrt{\varepsilon}})} \| 1 \|_{L^2(Q_T^{\sqrt{\varepsilon}})} \\
&\leq \liminf_{\varepsilon\to 0} D_4^2 \left| Q_T^{\sqrt{\varepsilon}} \right|^{\frac{1}{2}} \leq \lim_{\varepsilon\to 0} D_4^2 C_1^{\frac{1}{2}} e^{-\frac{1}{2\sqrt{\varepsilon}}} = 0,
\end{aligned}$$

and so $|\nabla u| \leq F(\theta)$ a.e. in Q_T, which means that $u(t) \in \mathbb{K}_{F(\theta(t))}$ for a.e. $t \in]0, T[$.

In particular, as $\theta \in L^\infty(Q_T)$, $u \in L^\infty(0, T; W^1_{\infty,0}(\Omega))$.

Step 2: $\partial_t u \in \left(L^\infty(0, T; W^1_{\infty,0}(\Omega))\right)'$.

Let $X = L^\infty(0, T; W^1_{\infty,0}(\Omega))$. Multiply the equation (13) by $\varphi \in X$ and integrate to obtain

$$\int_{Q_T} \partial_t u^\varepsilon \varphi + \int_{Q_T} k_\varepsilon(|\nabla u^\varepsilon|^2 - F_\varepsilon^2(\theta^\varepsilon))\, \nabla u^\varepsilon \cdot \nabla \varphi = \int_{Q_T} f^\varepsilon \varphi.$$

Then

$$\begin{aligned}
\left| \int_{Q_T} \partial_t u^\varepsilon \varphi \right| &\leq \|f^\varepsilon\|_{L^1(Q_T)} \|\varphi\|_{L^\infty(Q_T)} + \int_{Q_T} k_\varepsilon(|\nabla u^\varepsilon|^2 - F_\varepsilon^2(\theta^\varepsilon))\, |\nabla u^\varepsilon|\, |\nabla \varphi| \\
&\leq C\, \|f\|_{L^1(Q_T)} \|\varphi\|_X + \|\varphi\|_X \int_{Q_T} k_\varepsilon(|\nabla u^\varepsilon|^2 - F_\varepsilon^2(\theta^\varepsilon))\, |\nabla u^\varepsilon| \\
&= \left(C\, \|f\|_{L^1(Q_T)} + C_3\right) \|\varphi\|_X, \quad \text{by the previous Lemma.}
\end{aligned}$$

This proves that $\partial_t u^\varepsilon \in X'$ and that $\|\partial_t u^\varepsilon\|_{X'} \leq C\, \|f\|_{L^1(Q_T)} + C_3$. Since the disks in X' are compact sets for the weak-$*$ topology $\sigma(X', X)$, then a subsequence of $\{\partial_t u^\varepsilon\}_{0<\varepsilon<1}$ converges to $\partial_t u$ in X' endowed with this topology.

Step 3: (u, θ) satisfies the integral inequality in (8).

By density, we only need to prove the inequality for $v \in C^1([0, T]; H^1_0(\Omega))$ such that $v(0) = u_0$ and $v(t) \in \mathbb{K}_{F(\theta(t))}$, for a.e. $t \in]0, T[$. For such v, let $v_\lambda = u + \lambda(v - u)$, for $0 < \lambda < 1$. Consider

$$v_\lambda^\varepsilon = \frac{m}{m + A_\varepsilon}\, v_\lambda, \qquad A_\varepsilon = \|F^\varepsilon(\theta^\varepsilon) - F(\theta)\|_{C(\overline{Q}_T)}, \quad m \text{ as in Lemma 3.1.} \tag{19}$$

Note that, for a.e. $t \in [0, T]$, $v_\lambda(t) \in \mathbb{K}_{F(\theta(t))}$ and then $v_\lambda^\varepsilon(t) \in \mathbb{K}_{F^\varepsilon(\theta^\varepsilon(t))}$, as

$$|\nabla v_\lambda^\varepsilon(t)| \;=\; \frac{m}{m + A_\varepsilon} |\nabla v_\lambda| \;\leq\; \frac{m}{m + A_\varepsilon} F(\theta(t)) \;\leq\; F^\varepsilon(\theta^\varepsilon(t)),$$

being the last inequality true because it is equivalent to $\frac{F(\theta) - F^\varepsilon(\theta^\varepsilon)}{F^\varepsilon(\theta^\varepsilon)} \leq \frac{A_\varepsilon}{m}$.

Multiplying the equation of the problem (13) by $v_\lambda^\varepsilon - u^\varepsilon$ and integrating over Q_T we obtain

$$\int_{Q_T} \partial_t u^\varepsilon (v_\lambda^\varepsilon - u^\varepsilon) + \int_{Q_T} k_\varepsilon(|\nabla u^\varepsilon|^2 - F_\varepsilon^2(\theta^\varepsilon))\nabla u^\varepsilon \cdot \nabla(v_\lambda^\varepsilon - u^\varepsilon) = \int_{Q_T} f_\varepsilon(v_\lambda^\varepsilon - u^\varepsilon).$$

Since $v_\lambda^\varepsilon(t) \in \mathbb{K}_{F_\varepsilon(\theta^\varepsilon(t))}$, for a.e. t, and so $k_\varepsilon(|\nabla v_\lambda^\varepsilon|^2 - F_\varepsilon^2(\theta^\varepsilon)) = 1$, we get, using (12),

$$\int_{Q_T} \partial_t u^\varepsilon (v_\lambda^\varepsilon - u^\varepsilon) + \int_{Q_T} \nabla v_\lambda^\varepsilon \cdot \nabla(v_\lambda^\varepsilon - u^\varepsilon) \geq \int_{Q_T} f_\varepsilon(v_\lambda^\varepsilon - u^\varepsilon). \tag{20}$$

But

$$\begin{aligned}\int_{Q_T} \partial_t u^\varepsilon (v_\lambda^\varepsilon - u^\varepsilon) &= \int_{Q_T} \partial_t (u^\varepsilon - v_\lambda^\varepsilon)\ (v_\lambda^\varepsilon - u^\varepsilon) + \int_{Q_T} \partial_t v_\lambda^\varepsilon (v_\lambda^\varepsilon - u^\varepsilon)\\ &= \frac{1}{2}\int_\Omega \left(u_0^\varepsilon - \frac{m}{m+A_\varepsilon}u_0\right)^2 - \frac{1}{2}\int_\Omega (u^\varepsilon(T) - v_\lambda^\varepsilon(T))^2 + \int_{Q_T} \partial_t v_\lambda^\varepsilon (v_\lambda^\varepsilon - u^\varepsilon)\end{aligned}$$

and then, from (20),

$$\frac{1}{2}\int_\Omega \left(u_0^\varepsilon - \frac{m}{m+A_\varepsilon}u_0\right)^2 + \int_{Q_T} \partial_t v_\lambda^\varepsilon (v_\lambda^\varepsilon - u^\varepsilon) + \int_{Q_T} \nabla v_\lambda^\varepsilon \cdot \nabla(v_\lambda^\varepsilon - u^\varepsilon) \geq \int_{Q_T} f_\varepsilon(v_\lambda^\varepsilon - u^\varepsilon)$$

Letting $\varepsilon \to 0$, we obtain

$$\int_{Q_T} \partial_t v_\lambda (v_\lambda - u) + \int_{Q_T} \nabla v_\lambda \cdot \nabla(v_\lambda - u) \geq \int_{Q_T} f(v_\lambda - u)$$

or equivalently

$$\begin{aligned}\int_{Q_T} \lambda \partial_t u (v-u) + \lambda^2 \int_{Q_T} \partial_t (v-u)(v-u) + \lambda \int_{Q_T} \nabla u \cdot \nabla(v-u)&\\ + \lambda^2 \int_{Q_T} \nabla(v-u) \cdot \nabla(v-u) \geq \lambda \int_{Q_T} f(v-u)&\end{aligned} \tag{21}$$

and, dividing by λ and letting λ tend to 0,

$$\int_{Q_T} \partial_t u (v-u) + \int_{Q_T} \nabla u \cdot \nabla(v-u) \geq \int_{Q_T} f(v-u), \tag{22}$$

as we wanted to prove.

Let us prove now that (u, θ) solves (7)-(6).

Since $-\nabla \cdot (k_\varepsilon(|\nabla u^\varepsilon|^2 - F_\varepsilon^2(\theta^\varepsilon))\nabla u^\varepsilon)$ is uniformly bounded in X', because it is equal to $f^\varepsilon - \partial_t u^\varepsilon$, we have

$$-\nabla \cdot (k_\varepsilon(|\nabla u^\varepsilon|^2 - F_\varepsilon^2(\theta^\varepsilon))\nabla u^\varepsilon) \longrightarrow \chi \quad \text{in } X' \text{ for the weak -}* \text{ topology } \sigma(X', X).$$

Let v be any function verifying:

· $v \in L^2(0,T;H_0^1(\Omega))$ such that $\partial_t v \in L^2(0,T;H^{-1}(\Omega))$;

· $v(t) \in \mathbb{K}_{F(\theta(t))}$, for a.e. $t \in]0,T[$, $v(0) = u_0$.

Multiply the first equation of the system (11) by $v - u^\varepsilon$ and integrate over Q_T. Then

$$\int_{Q_T} \partial_t u^\varepsilon (v - u^\varepsilon) + \int_{Q_T} k_\varepsilon(|\nabla u^\varepsilon|^2 - F_\varepsilon^2(\theta^\varepsilon))\nabla u^\varepsilon \cdot \nabla(v - u^\varepsilon) = \int_{Q_T} f^\varepsilon(v - u^\varepsilon). \qquad (23)$$

Observe that

$$\begin{aligned}\int_{Q_T} \partial_t u^\varepsilon (v - u^\varepsilon) &= \int_{Q_T} \partial_t(u^\varepsilon - v)(v - u^\varepsilon) + \int_{Q_T} \partial_t v(v - u^\varepsilon)\\ &= -\frac{1}{2}\int_\Omega (u^\varepsilon(T) - v(T))^2 + \frac{1}{2}\int_\Omega (u_0^\varepsilon - u_0)^2 + \int_{Q_T} \partial_t v(v - u^\varepsilon)\\ &\le \frac{1}{2}\int_\Omega (u_0^\varepsilon - u_0)^2 + \int_{Q_T} \partial_t v(v - u^\varepsilon). \qquad (24)\end{aligned}$$

Define $v^\varepsilon(t) = \frac{m}{m+A_\varepsilon}u$ as in (19) and notice that v^ε converges strongly to u in X and $v^\varepsilon(t) \in \mathbb{K}_{F_\varepsilon(\theta^\varepsilon(t))}$ for a.e. $t \in]0,T[$.

So, using (23) and (24), we obtain

$$\begin{aligned}&\int_{Q_T} \partial_t v(v - u^\varepsilon) + \frac{1}{2}\int_\Omega (u_0^\varepsilon - u_0)^2\\ &\qquad + \int_{Q_T} k_\varepsilon(|\nabla u^\varepsilon|^2 - F_\varepsilon^2(\theta^\varepsilon))\nabla u^\varepsilon \cdot \nabla((v - v^\varepsilon) + (v^\varepsilon - u^\varepsilon)) \ge \int_{Q_T} f^\varepsilon(v - u^\varepsilon).\end{aligned}$$

But

$$\begin{aligned}&\int_{Q_T} k_\varepsilon(|\nabla u^\varepsilon|^2 - F_\varepsilon^2(\theta^\varepsilon))\nabla u^\varepsilon \cdot \nabla((v - v^\varepsilon) + (v^\varepsilon - u^\varepsilon))\\ &= \int_{Q_T} k_\varepsilon(|\nabla u^\varepsilon|^2 - F_\varepsilon^2(\theta^\varepsilon))\nabla u^\varepsilon \cdot \nabla(v - v^\varepsilon) + \int_{Q_T} k_\varepsilon(|\nabla u^\varepsilon|^2 - F_\varepsilon^2(\theta^\varepsilon))\nabla u^\varepsilon \cdot \nabla(v^\varepsilon - u^\varepsilon)\\ &\qquad \le \int_{Q_T} k_\varepsilon(|\nabla u^\varepsilon|^2 - F_\varepsilon^2(\theta^\varepsilon))\nabla u^\varepsilon \cdot \nabla(v - v^\varepsilon) + \int_{Q_T} \nabla v^\varepsilon \cdot \nabla(v^\varepsilon - u^\varepsilon),\end{aligned}$$

using (12), noticing that $v^\varepsilon(t) \in \mathbb{K}_{F_\varepsilon(\theta^\varepsilon(t))}$, for a.e. $t \in]0,T[$. So

$$\begin{aligned}&\int_{Q_T} \partial_t v(v - u^\varepsilon) + \frac{1}{2}\int_\Omega (u_0^\varepsilon - u_0)^2 + \int_{Q_T} -\nabla \cdot \left(k_\varepsilon(|\nabla u^\varepsilon|^2 - F_\varepsilon^2(\theta^\varepsilon))\nabla u^\varepsilon\right)(v - v^\varepsilon)\\ &\qquad + \int_{Q_T} \nabla v^\varepsilon \cdot \nabla(v^\varepsilon - u^\varepsilon) \ge \int_{Q_T} f^\varepsilon(v - u^\varepsilon).\end{aligned}$$

Recalling the strong convergence of v^ε to u in X and letting $\varepsilon \to 0$ in the above inequality, we obtain

$$\int_{Q_T} \partial_t v(v - u) + \int_{Q_T} \chi(v - u) \ge \int_{Q_T} f(v - u). \qquad (25)$$

Multiplying the first equation of the system (11) by $v-u$ and integrating (in the duality sense) in Q_T, we get, letting $\varepsilon \to 0$,

$$\int_{Q_T} \partial_t u(v-u) + \int_{Q_T} \chi(v-u) = \int_{Q_T} f(v-u). \tag{26}$$

From (25) and (26), we deduce that

$$\int_{Q_T} \partial_t (v-u)(v-u) \geq 0. \tag{27}$$

Finally, we have

$$\begin{aligned} \int_{Q_T} \partial_t v(v-u) + \int_{Q_T} \nabla u \cdot \nabla (v-u) &= \int_{Q_T} \partial_t u(v-u) + \int_{Q_T} \partial_t (v-u)(v-u) \\ &\quad + \int_{Q_T} \nabla u \cdot \nabla (v-u) \\ &\geq \int_{Q_T} \partial_t u(v-u) + \int_{Q_T} \nabla u \cdot \nabla (v-u) \qquad \text{by (27)} \\ &\geq \int_{Q_T} f(v-u), \qquad \text{by (22)} \end{aligned}$$

and the proof of (7) is complete. □

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